TECH SERIES

Performance Analysis of the Ideal Rocket Motor

Foreword, *by Charles E. Rogers*

One of the fundamental equations in the field of rocketry is the equation for the thrust of an ideal rocket motor. This equation can be used to predict the thrust of solid rocket motors and liquid rocket engines as a function of chamber pressure, ratio of specific heats for the flow through the nozzle, nozzle throat area and expansion ratio, and atmospheric pressure.

$$F = p_c A_{th} \left\{ \frac{2\gamma^2}{\gamma - 1} \left(\frac{2}{\gamma + 1} \right)^{\frac{\gamma+1}{\gamma-1}} \left[1 - \left(\frac{p_e}{p_c} \right)^{\frac{\gamma-1}{\gamma}} \right] \right\}^{\frac{1}{2}} + (p_e - p_\infty) A_e$$

Where:

A_e = nozzle exit area

A_{th} = nozzle throat area

F = thrust

p_c = chamber pressure

p_e = nozzle exit pressure

p_∞ = atmospheric pressure

γ = ratio of specific heats for flow through nozzle

Despite being based on an ideal perfect gas analysis, the ideal rocket motor thrust equation is accurate to within 1-6% for most rocket motors, and can be accurate to within 1-3% when a non-ideal correction for nozzle divergence angle is included. The fact that the thrust of a rocket motor increases with altitude, and that maximum thrust is achieved by using an optimum expansion ratio can both be derived using this equation. All propellant performance analysis programs, rocket engine analysis programs, and solid rocket motor simulation programs use this equation in some form, making it part of the foundation of rocket performance analysis.

The first individual to apply the theory of gas flow through nozzles to rocket engines, the central theory for the derivation of the ideal rocket motor thrust equation, was Hermann Oberth. His original derivation culminated in the Oberth exhaust velocity equation, named in his honor.

$$V_e = \sqrt{\frac{2\gamma}{(\gamma-1)} \frac{RT_c}{M_f} \left[1 - \left(\frac{p_e}{p_c} \right)^{\frac{\gamma-1}{\gamma}} \right]}$$

Where:

M_f = molecular weight of gas

R = universal gas constant

T_c = combustion temperature

V_e = exhaust velocity

Oberth submitted his derivation of the Oberth exhaust velocity equation as his doctoral thesis at Heidelburg University, but it was rejected. At great personal expense he published his thesis in 1923 as a pamphlet of less than 100 pages titled *Die Rakete zu den Planetenraumen* (The Rocket into Interplanetary Space). In this document Oberth proved theoretically that the thrust of a rocket increases with altitude, and that a rocket produces maximum thrust in a vacuum. Oberth cited as experimental evidence a remarkable series of experiments performed by Robert H. Goddard in 1915-1916 that showed increased thrust and efficiency with altitude. Oberth's theoretical analysis and Goddard's experiments proved that a rocket could produce thrust in a vacuum. These results showed that the rocket was a practical means of propulsion in space, making spaceflight possible.

In the author's opinion one of the best technical articles ever written covering the derivation of the ideal rocket motor thrust equation was by Martin Summerfield and published in 1959 in *Volume 12 - Jet Propulsion Engines* of the Princeton *High Speed Aerodynamics and Jet Propulsion* series. The full derivation of the ideal rocket motor thrust equation, specific impulse, characteristic velocity, nozzle exit pressure as a function of nozzle expansion ratio, non-ideal corrections for nozzle divergence angle, flow separation in overexpanded conical nozzles, all are covered in extensive detail in Summerfield's technical article. After nearly 30 years the fundamental equations and the clarity of the technical writing have withstood the test of time. This 12 volume series was published by the Princeton University Press under contract to the United States Government. Under that contract after ten years the material entered the public domain, and the author and *High Power Rocketry* are pleased to republish this material for a new generation of rocketeers.

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SECTION G

THE LIQUID PROPELLANT ROCKET ENGINE

MARTIN SUMMERFIELD

G,1. Introduction. The scientific development of the modern rocket engine has been made possible to a great extent by the results of research in the fields of chemical physics, high temperature combustion, high intensity heat transfer, gas dynamics, and heat-resistant materials. Much of this work has taken place in recent years and has not been incorporated heretofore in the standard textbook literature in a form suitable for systematic study. For an up-to-date comprehensive treatment of the liquid propellant rocket engine, therefore, it would have been necessary to introduce in this section all of the new developments mentioned. However, in the planning of this series, many of these topics were placed in other sections to allow a more logical arrangement of the entire subject matter. Therefore, in this section, the pertinent results or conclusions derived from these modern investigations are merely stated in their simplest form, and the reader is directed by cross references to the other volumes where the complete treatments can be found.

In rocketry, as in any field that has grown rapidly, the terminology has not yet been fully standardized and accepted. Therefore it is appropriate to start with a few definitions.

Rocket propulsion is a system of propulsion that depends on forward thrust created by rearward ejection of a fluid jet through a nozzle mounted in the vehicle, with the special condition that the fluid in the jet originate entirely from tanks within the vehicle. It is this special condition that distinguishes the rocket from other classes of jet engines that ingest the surrounding medium (air or water) to form the driving jet. Therefore the rocket engine is able to function not only under the usual conditions of flight through the atmosphere (or under water), but in the vacuum outside the atmosphere.

A *rocket propellant* is the fluid substance that forms the driving jet, although the term is used most frequently to refer to the driving fluid in its chemical state before combustion. The term is used also to denote *one* of the reactants in a multicomponent propellant system. A *fuel* is any propellant that can burn in the presence of oxygen, and the term includes not only hydrocarbons but other substances (e.g. ammonia, powdered aluminum) as well. An *oxidizer* is a propellant that can support the combustion of a fuel, and is applicable to substances that may not contain oxygen (e.g. fluorine) as well as those that do. A *bipropellant system* is one that consists of two reactants, usually a fuel and an oxidizer. A *monopropellant* is a single substance that can be caused to react in the combustion chamber to generate hot gas to form the driving jet. This term applies strictly only to a single compound (e.g. ethylene oxide) that undergoes a decomposition reaction, but it has been applied also to a propellant mixture that is stored in a single propellant tank (e.g. a mixture of methyl nitrate and methyl alcohol).

The *rocket motor* is usually understood to be the part of the engine in which the propellants are burned and the jet is formed, while the term *rocket engine* usually refers to the entire propulsion system, including the tanks if they are constructed integrally with the engine. The term *thrust cylinder* has been used in place of rocket motor in some writings, but the latter term is historically the oldest and the most widely preferred. In the conventional solid propellant rocket, the engine and the motor happen to be the same piece of apparatus because the propellant is stored in the combustion chamber of the rocket motor; this is not so in the liquid propellant rocket.

G,2. Performance Analysis of the Ideal Rocket Motor. The performance analysis of a rocket motor comprises calculations of the thrust F, the effective exhaust velocity c, the adiabatic combustion temperature in the chamber T_c, the thrust coefficient C_F, the characteristic velocity c^*, and certain efficiencies η. Performance parameters derived from c are the specific impulse I_{sp} and the specific propellant consumption w_{sp}.

The thrust equation is the fundamental starting point. In general, the thrust exerted on a duct of arbitrary shape can be calculated from the momentum equation written in integrated form appropriate for one-dimensional flow problems. (See III,B.) Let $\dot{m}_i$ be the rate of mass flow into the inlet, p_i the static pressure at the inlet, V_i the stream velocity at the inlet, and A_i the area of the inlet, and let the corresponding quantities at the exit be indicated by the subscript $_e$. Then,

$$\text{The stream thrust at the inlet} = (\dot{m}_i V_i + p_i A_i)$$

$$\text{The stream thrust at the exit} = (\dot{m}_e V_e + p_e A_e)$$

$$\text{The total force on the external duct surface} = F + p_\infty(A_e - A_i)$$

$$F + p_\infty(A_e - A_i) = (\dot{m}_e V_e + p_e A_e) - (\dot{m}_i w_i + p_i A_i)$$

The external force on the duct is expressed here as if the pressure on the external surface were identical with the ambient pressure p_∞ of the atmosphere, although in actual flight this is not so. Therefore, for a duct in flight, this equation implies a certain arbitrary separation between the thrust F and the aerodynamic drag D. Separation in this manner is justified by its convenience, because the thrust measured in a ground test of the propulsion system is closely equal to the thrust F thus calculated.

In the particular case of a rocket, $\dot{m}_i$ and A_i may be set equal to zero. Then,

$$F = \dot{m}V_e + (p_e - p_\infty)A_e \tag{2-1}$$

With reasonably well-designed exhaust nozzles, the exit pressure p_e is nearly or exactly equal to the ambient pressure p_∞, so that the second term is in the nature of a small correction to the thrust. This makes it convenient to define an *effective exhaust velocity* c such that

$$F = \dot{m}c \tag{2-2}$$

$$c = V_e + \frac{(p_e - p_\infty)A_e}{\dot{m}} \tag{2-3}$$

Clearly, c equals V_e if the nozzle is designed properly, that is, if the p_e equals p_∞. For any given values of chamber pressure p_c and of p_∞ and $\dot{m}$, both F and c reach their maximum values when the exit area A_e of the nozzle is chosen to produce a static pressure p_e at the exit exactly equal to p_∞. (This will be proved later.) The exit velocity and the correction term $(p_e - p_\infty)A_e/\dot{m}$ both vary strongly with A_e, but in opposite directions so that the sum is quite insensitive to A_e, that is, the maximum is very flat. As a result, the effective exhaust velocity measured by the ratio of F to $\dot{m}$ can be taken to be the value of $(V_e)_{\text{opt}}$, even if the actual nozzle used in the test is somewhat off-design. Herein lies the practical significance of the concept of the effective exhaust velocity (see Fig. G,2a.)

The specific propellant consumption w_{sp}, defined as the weight rate of consumption of propellant $\dot{w}$ per unit thrust, is another useful index of rocket engine performance. Let g_0 denote the standard acceleration due to gravity.

$$w_{\text{sp}} = \frac{\dot{w}}{F} = \frac{\dot{m}g_0}{F} = \frac{g_0}{c} \tag{2-4}$$

The specific impulse I_{sp} (called specific thrust in some writings) is defined as the propulsive impulse delivered by the engine per unit weight of propellant.

$$I_{\text{sp}} = \frac{Fdt}{dw} = \frac{F}{\dot{w}} = \frac{1}{w_{\text{sp}}} = \frac{c}{g_0} \tag{2-5}$$

A figure of merit that is sometimes quoted is the impulse-weight ratio, I/W, of a loaded rocket propulsion system or of a loaded rocket vehicle.

If the firing program calls for constant pressure and hence constant specific impulse,

$$\frac{I}{W} = I_{sp}\nu \tag{2-6}$$

where ν is the *propellant loading fraction,* that is, the ratio of the initial mass of propellant when the rocket is fully loaded to the gross mass of the loaded rocket. Obviously, ν approaches unity as the structural effectiveness is improved. Therefore, I/W measures both the performance of the engine and the effectiveness of the structure of the rocket.

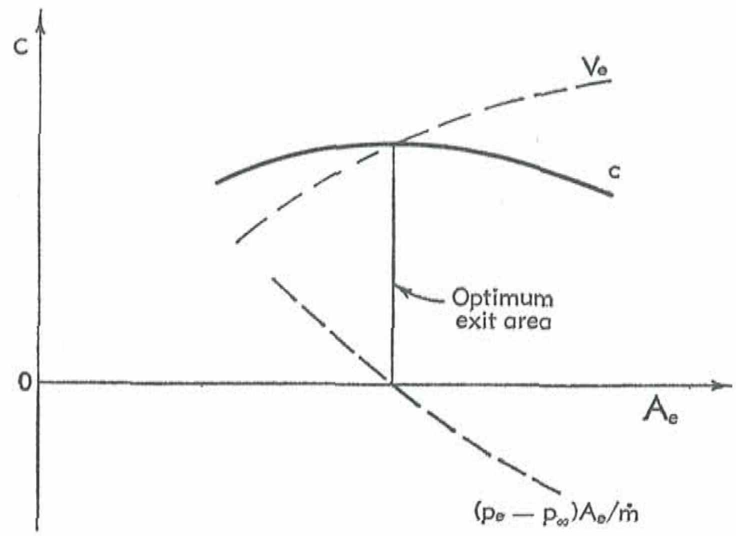

Fig. G,2a. Variation of effective exhaust velocity with exit area.

The matter of units of specific impulse deserves comment. The definition involves the mass rate of flow of propellant expressed in weight units, so that in the fps system I_{sp} should be lb sec/lb. It is common practice to denote this ratio simply as sec, canceling the lb in numerator and denominator, disregarding the fact that one lb refers to a force and the other to a mass. That this is an error becomes apparent when it is realized that the specific impulse has the dimensions of a velocity.

In order to develop several additional performance parameters, it is necessary to describe in detail the thermodynamic and gas dynamic processes in the rocket motor. As a starting point, it is a great simplification to deal with the so-called *ideal rocket motor*. From a practical standpoint, the ideal rocket motor is a useful concept because it leads to simple theoretical formulas for F, c, T_c, C_F, and c^*, which otherwise would have to be presented in tables or in graphs.

The ideal rocket motor analysis rests on the following simplifications: (1) the propellant gas obeys the perfect gas law; (2) its specific heat is constant, independent of temperature; (3) the flow is parallel to the axis of the motor and uniform in every plane normal to the axis, thus constituting a one-dimensional problem; (4) there is no frictional dissipation in the chamber or nozzle; (5) there is no heat transfer to the motor walls; (6) the flow velocity in the chamber before the nozzle entrance is zero; (7) combustion or heat addition is completed in the chamber at constant pressure and does not occur in the nozzle; and (8) the process is steady in time.

The thermodynamic process is indicated in Fig. G,2b, both in the p, V diagram and in the h, s diagram. Combustion at constant pressure

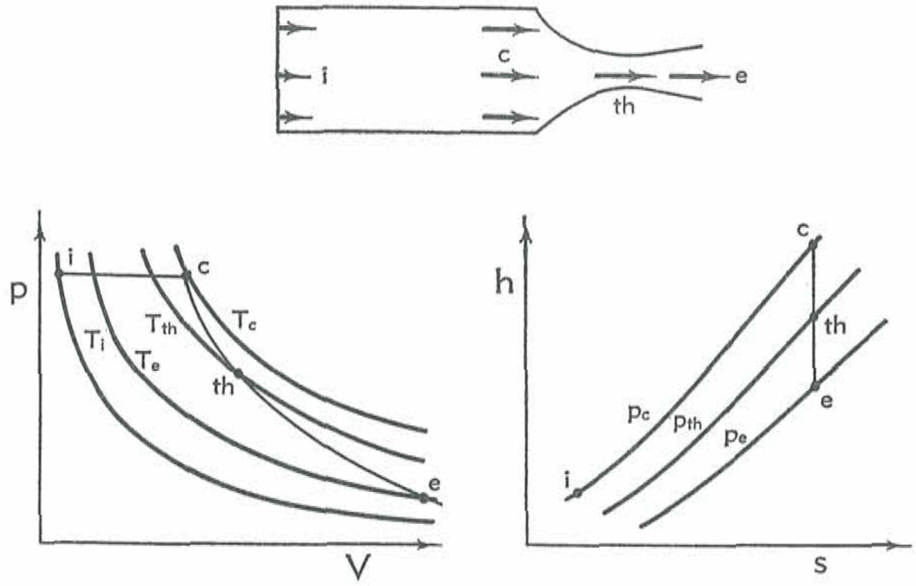

Fig. G,2b. Ideal thermodynamic processes in the combustion chamber and nozzle of a rocket motor.

moves the state point from i to c. The highest temperature occurs at c. Then, since the frictionless and adiabatic conditions assumed for the ideal rocket motor imply an isentropic expansion in the nozzle (see III,B for proof; also I,A), the state point moves along the constant entropy line from c to e during the expansion process. As indicated in the motor sketch of Fig. G,2b, the subscripts i, c, th, and e employed in the following analysis refer respectively to the unburned state at the injector, the all-burned state in the chamber just before expansion, the state at the throat of the de Laval nozzle, and the state at the exit of the nozzle.

The combustion temperature (or adiabatic flame temperature) T_c is determined by the heat of combustion at constant pressure per unit mass Δh_c.

$$\Delta h_c = c_p(T_c - T_i) \tag{2-7}$$

At any station in the nozzle, the entropy, pressure, temperature, velocity, and Mach number are given by the following relations (III,B):

$$p = \rho \frac{R}{\mathfrak{M}} T; \quad c_p - c_v = \frac{R}{\mathfrak{M}}; \quad c_p = \frac{\gamma}{\gamma - 1} \frac{R}{\mathfrak{M}} \tag{2-8}$$

$$s = s_c; \quad \frac{T}{T_c} = \left(\frac{p}{p_c}\right)^{\frac{\gamma-1}{\gamma}} \tag{2-9}$$

$$\left.\begin{aligned} \tfrac{1}{2}V^2 &= c_p(T_c - T) = \frac{\gamma}{\gamma - 1} \frac{RT_c}{\mathfrak{M}} \left[1 - \left(\frac{p}{p_c}\right)^{\frac{\gamma-1}{\gamma}}\right] \\ \frac{A}{A_{\text{th}}} &= \frac{1}{M} \left[\frac{1 + \frac{\gamma - 1}{2} M^2}{\frac{\gamma + 1}{2}}\right]^{\frac{\gamma+1}{2(\gamma-1)}} \end{aligned}\right\} \tag{2-10}$$

Since the over-all pressure ratio p_c/p_∞ is always sufficiently large in rockets to establish sonic flow at the throat, then

$$M_{\text{th}} = \frac{V_{\text{th}}}{a_{\text{th}}} = 1; \quad V_{\text{th}} = \left(\gamma \frac{R}{\mathfrak{M}} T_{\text{th}}\right)^{\frac{1}{2}} \tag{2-11}$$

$$\frac{p_{\text{th}}}{p_c} = \left(\frac{2}{\gamma + 1}\right)^{\frac{\gamma}{\gamma-1}}; \quad \frac{T_{\text{th}}}{T_c} = \frac{2}{\gamma + 1} \tag{2-12}$$

The specific heat ratios of typical rocket jet gases range between 1.1 and 1.3. The first figure corresponds to mixtures at very high temperatures with large concentrations of water vapor and large effective specific heats due to strong dissociation; the latter figure applies to moderate temperature gases with moderate concentrations of H_2O and CO_2. With $\gamma = 1.2$, it can be seen that the drop in pressure from the chamber to the throat is nearly half the chamber pressure, while the drop in temperature is only about one tenth the chamber temperature.

The mass flow through the nozzle can be expressed in terms of the flow conditions at any station. Let A be the cross-sectional area at any station:

$$\dot{m} = \rho V A = p_c A \left\{\frac{2\gamma}{\gamma - 1} \frac{\mathfrak{M}}{RT_c} \left(\frac{p}{p_c}\right)^{\frac{2}{\gamma}} \left[1 - \left(\frac{p}{p_c}\right)^{\frac{\gamma-1}{\gamma}}\right]\right\}^{\frac{1}{2}} \tag{2-13}$$

A plot of mass flow per unit area $\dot{m}/A$ against static pressure ratio p/p_c, shown in Fig. G,2c, exhibits a maximum at the throat of the de Laval nozzle, just as expected.

$$\frac{\dot{m}}{A_{\text{th}}} = p_c \left\{\gamma \left(\frac{2}{\gamma + 1}\right)^{\frac{\gamma+1}{\gamma-1}} \frac{\mathfrak{M}}{RT_c}\right\}^{\frac{1}{2}} \tag{2-14}$$

By equating the two expressions, Eq. 2-13 and 2-14, for $\dot{m}$, an expression for A/A_{th} is obtained.

$$\frac{A}{A_{th}} = \frac{\left(\frac{\gamma - 1}{2}\right)^{\frac{1}{2}} \left(\frac{2}{\gamma + 1}\right)^{\frac{\gamma+1}{2(\gamma-1)}}}{\left(\frac{p}{p_c}\right)^{\frac{1}{\gamma}} \left[1 - \left(\frac{p}{p_c}\right)^{\frac{\gamma-1}{\gamma}}\right]^{\frac{1}{2}}} \tag{2-15}$$

By inserting A_e and p_e for A and p in Eq. 2-15, the nozzle area ratio ϵ ($\equiv A_e/A_{th}$) can be expressed as a function of p_c/p_e. This relation is plotted in Fig. G,2d for several values of γ.

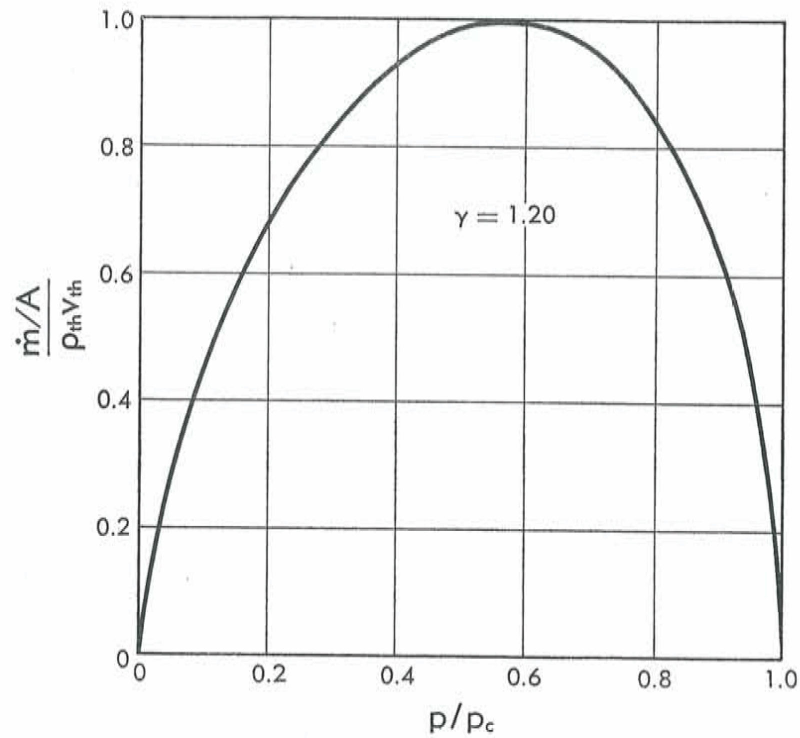

Fig. G,2c. Variation of mass flow per unit area $\dot{m}/A$ with pressure ratio p/p_c.

The maximum exit velocity $(V_e)_{max}$, obtained by setting p_e/p_c in Eq. 2-10 equal to zero, is

$$(V_e)_{max} = \sqrt{\frac{2\gamma}{\gamma - 1} \frac{RT_c}{\mathfrak{M}}} \tag{2-16}$$

It is interesting that the maximum exit velocity, obtained by expansion to zero pressure, is greater than the root-mean-square molecular velocity in the chamber by the factor $[2\gamma/3(\gamma - 1)]^{\frac{1}{2}}$, or about 2 for $\gamma = 1.2$. This result follows directly, of course, from molecular energy considerations.

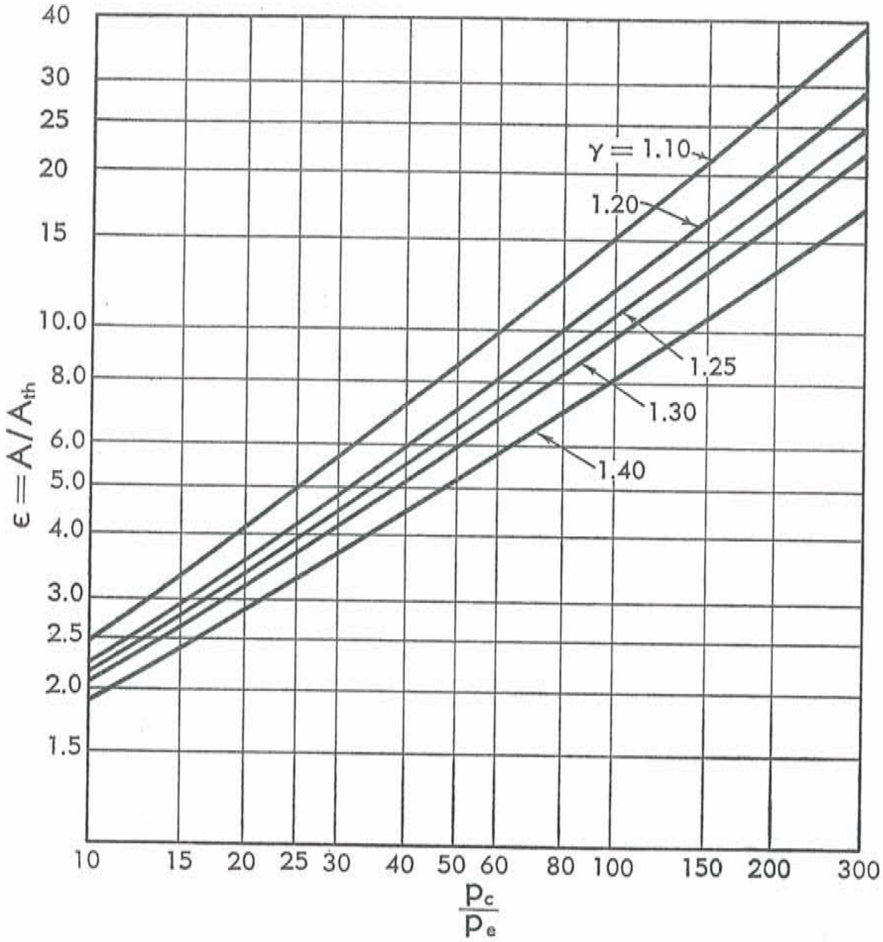

Fig. G,2d. Variation of nozzle area ratio with pressure ratio. After Sutton, G.P., *Rocket Propulsion Elements*, Wiley, 1956, by permission.

The thrust formula (Eq. 2-1) can now be expressed in terms of the pressures by substituting Eq. 2-10 for V_e and Eq. 2-14 for $\dot{m}$.

$$F = p_c A_{\text{th}} \left\{ \frac{2\gamma^2}{\gamma - 1} \left(\frac{2}{\gamma + 1} \right)^{\frac{\gamma+1}{\gamma-1}} \left[1 - \left(\frac{p_e}{p_c} \right)^{\frac{\gamma-1}{\gamma}} \right] \right\}^{\frac{1}{2}} + (p_e - p_\infty) A_e \quad (2\text{-}17)$$

From this formula, it is clear that the thrust does not depend at all on the combustion temperature T_c, but depends mainly on the dimensions of the nozzle A_e and A_{th}, and on the chamber pressure p_c.

An important performance parameter, the rocket thrust coefficient C_F, can now be deduced. The defining equation is

$$C_F \equiv \frac{F}{p_c A_{\text{th}}} \quad (2\text{-}18)$$

From Eq. 2-17 C_F can be evaluated:

$$C_F = \left\{ \frac{2\gamma^2}{\gamma - 1} \left(\frac{2}{\gamma + 1} \right)^{\frac{\gamma+1}{\gamma-1}} \left[1 - \left(\frac{p_e}{p_c} \right)^{\frac{\gamma-1}{\gamma}} \right] \right\}^{\frac{1}{2}} + \left(\frac{p_e}{p_c} - \frac{p_\infty}{p_c} \right) \epsilon \quad (2\text{-}19)$$

Since p_e/p_c is a function of ϵ according to Eq. 2-15, C_F depends only on the three independent variables γ, p_c/p_∞, and ϵ. Graphs of this function are presented in Fig. G,2e and G,2f for $\gamma = 1.2$ and $\gamma = 1.3$, respectively.

There are several features of these curves that deserve attention. First, each curve shows a maximum value of C_F at a certain area ratio which may, for this purpose, be called ϵ_{opt}. It can be shown analytically, by differentiating Eq. 2-19 and setting the derivative equal to zero, that the peak value occurs for a value of ϵ such that $p_e = p_\infty$, that is, for a

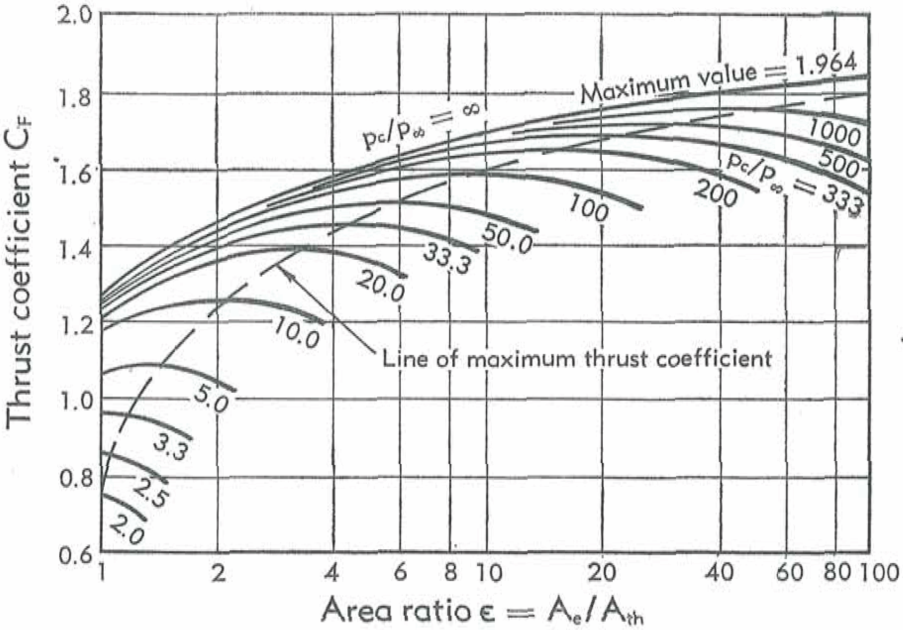

Fig. G,2e. Variation of rocket thrust coefficient with nozzle area ratio and pressure ratio p_c/p_∞ for $\gamma = 1.2$. After Sutton, G.P., *op. cit.*

properly expanded nozzle. The area ratio ϵ_{opt} for proper expansion can be determined from the peak of the appropriate curve in Fig. G,2e or G,2f, or, more accurately, from Fig. G,2d. A nozzle having an area ratio less than ϵ_{opt} is said to be *underexpanded,* and one having an area ratio more than ϵ_{opt} is *overexpanded.* Clearly, nozzles that are either overexpanded or underexpanded produce less thrust than a properly expanded nozzle. This conclusion can be proved in another way, by considering the distribution of pressure on the inner and outer surfaces of the rocket motor, as shown in Fig. G,2g. Downstream of the section indicated as ϵ_{opt}, the internal pressure is less than the external pressure, so that this portion of the cone acts to produce a force opposed to the thrust of the rocket motor as a whole. Therefore it is better to dispense with it and to terminate the nozzle at ϵ_{opt}. Consequently the highest thrust is produced with a properly expanded nozzle.

Examination of the pressure distribution pictured in Fig. G,2f shows that C_F must be somewhat greater than unity, except for the unusual case of low chamber pressure ($p_c/p_0 \cong 1$) and for possibly greatly overexpanded nozzles. Viewed in the simplest way, the rocket motor is a pressurized vessel with a hole of area A_{th} in the aft wall, and so it would be

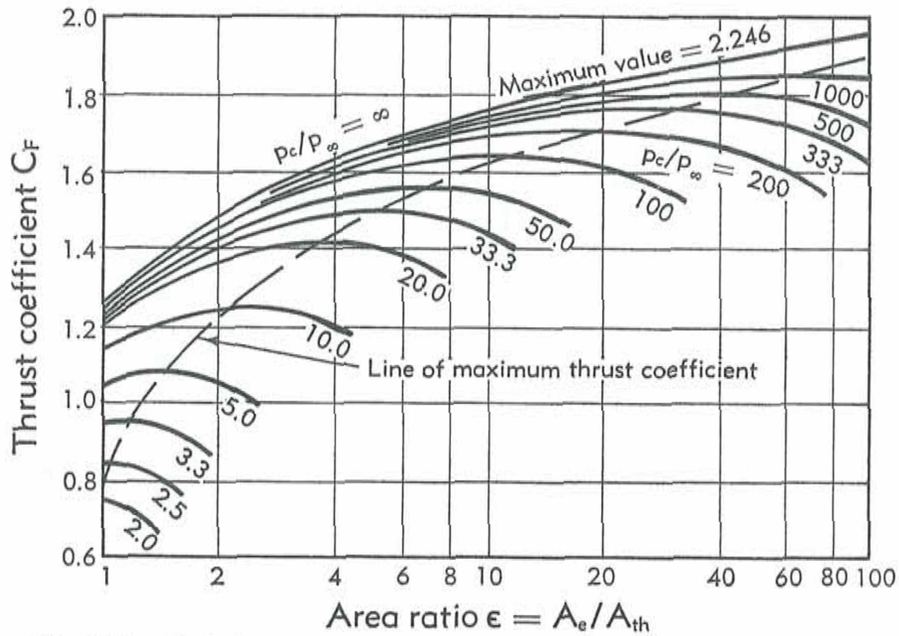

Fig. G,2f. Variation of rocket thrust coefficient with nozzle area ratio and pressure ratio p_c/p_∞ for $\gamma = 1.3$. After Sutton, G.P., *op. cit.*

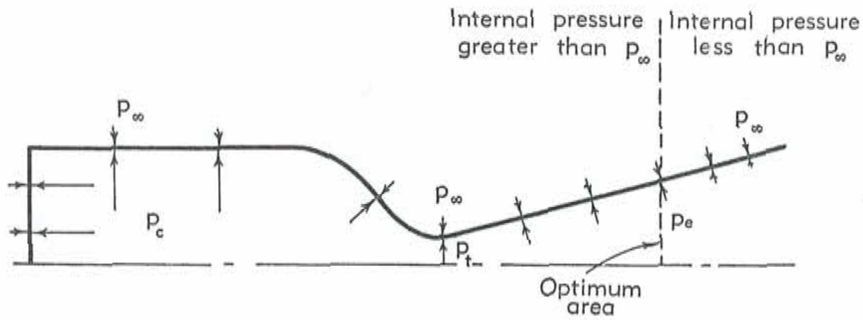

Fig. G,2g. Optimization of nozzle area ratios.

acted upon by an unbalanced force $(p_c - p_0)A_{th}$. To this must be added the additional unbalance due to the pressure reduction in the entrance cone of the nozzle. Then, if $p_0 \ll p_c$, the ratio F/p_cA_{th} must be somewhat larger than 1.

The maximum values of C_F on the families of curves in Fig. G,2e and G,2f can be connected by a smooth curve obtained by setting $p_e = p_\infty$

in Eq. 2-19.

$$(C_F)_{\max} = \left\{\frac{2\gamma^2}{\gamma - 1}\left(\frac{2}{\gamma + 1}\right)^{\frac{\gamma+1}{\gamma-1}}\left[1 - \left(\frac{p_\infty}{p_c}\right)^{\frac{\gamma-1}{\gamma}}\right]\right\}^{\frac{1}{2}} \tag{2-20}$$

The dependence on (p_∞/p_c) can be replaced by dependence on ϵ_{opt} through the relation (Eq. 2-15). The curve $(C_F)_{\max}$ vs. ϵ_{opt} itself reaches an ultimate value for infinite expansion. Thus, if (p_∞/p_c) is set equal to zero in Eq. 2-20,

$$(C_F)_{\text{ult}} = \left[\frac{2\gamma^2}{\gamma - 1}\left(\frac{2}{\gamma + 1}\right)^{\frac{\gamma+1}{\gamma-1}}\right]^{\frac{1}{2}} \tag{2-21}$$

For example, $(C_F)_{\text{ult}} = 2.246$ for $\gamma = 1.2$.

Inspection of Fig. G,2e and G,2f discloses that, for a prescribed nozzle area ratio ϵ, the thrust coefficient increases monotonically as p_c/p_∞ increases. This becomes clear when Eq. 2-19 is written in terms of the thrust coefficient for a given nozzle when it is operating in a vacuum:

$$C_F = (C_F)_{\text{vac}} - \epsilon\frac{p_\infty}{p_c} \tag{2-22}$$

Thus, for given ϵ and p_c, the increase in thrust with increasing p_c/p_∞ stems entirely from the reduction of the pressure acting on the external surfaces of the rocket motor.

It is significant that C_F is completely independent of combustion temperature T_c and of molecular weight $\mathfrak{M}$. Consequently, as a figure of merit, it is insensitive to the efficiency of combustion, but it is sensitive to the quality of the exhaust nozzle. In practice, the test engineer compares the measured C_F, computed from p_c, A_{th}, and F by means of Eq. 2-18, with the theoretical C_F computed from Eq. 2-19 to determine whether the nozzle is functioning properly, and in this way he can localize to some extent the cause of an unexpected defect in specific impulse. The other possible area for loss is in the combustion process. To detect combustion inefficiency, the performance parameter c^* is useful.

The characteristic velocity c^* is defined as follows:

$$c^* \equiv \frac{p_c A_{\text{th}}}{\dot{m}} \tag{2-23}$$

It follows immediately from Eq. 2-2 and 2-18 that

$$c = C_F c^* \tag{2-23a}$$

A theoretical expression for c^* is obtainable from Eq. 2-14.

$$c^* = \left[\frac{1}{\gamma}\left(\frac{\gamma + 1}{2}\right)^{\frac{\gamma+1}{\gamma-1}}\frac{RT_c}{\mathfrak{M}}\right]^{\frac{1}{2}} \tag{2-24}$$

From this formula it appears that c^* depends mainly on conditions in the combustion chamber, that is, on flame temperature and combustion product composition. Consequently, just as C_F is used as an index of the quality of the exhaust nozzle, so c^* is used in practice as an index of the efficiency of combustion. The test engineer determines c^* from measured values of p_c, A_{th}, $\dot{m}$, and compares it with the theoretical value (Eq. 2-24). In this way, a defect in specific impulse can be traced to a possible loss in the combustion process. Although the performance of a rocket motor is adequately described by the exhaust velocity c, which requires only the measurement of F and $\dot{m}$, it is the usual practice to measure at the same time p_c and A_{th} in order to compute C_F and c^* for diagnostic purposes.

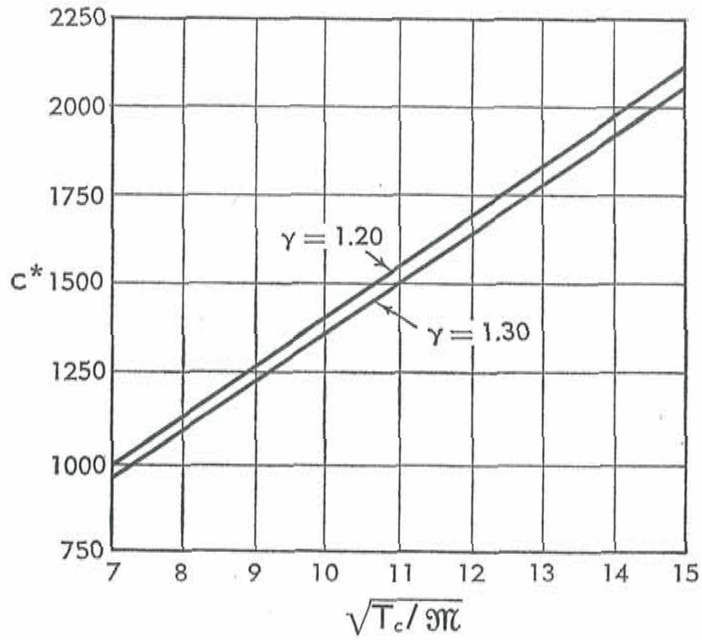

Fig. G,2h. Variation of characteristic velocity c^* with $\sqrt{T_c/\mathfrak{M}}$.

(Careful consideration of the flow process reveals that there exists some slight cross-dependence of C_F and c^*, that is, the former is slightly sensitive to the combustion process and the latter is somewhat affected by the flow conditions in the nozzle, but this is usually ignored.)

Curves of c^* vs. $(T_c/\mathfrak{M})^{\frac{1}{2}}$ for $\gamma = 1.2$ and $\gamma = 1.3$ are plotted in Fig. G,2h. It is significant that c^*, and therefore the specific impulse, depends as much on molecular weight $\mathfrak{M}$ as on flame temperature. Thus, it is just as important for the product gas to have a low mean molecular weight as a high temperature. This point will arise later when actual propellants are considered. It will be pointed out then that the optimum fuel-oxidizer mixture ratio is not necessarily the one that produces the highest flame temperature, and that a particular propellant combination may be very hot but no better than another much cooler one from the standpoint of specific impulse.

The magnitude of c^* is of interest. Because of its dependence on $(RT_c/\mathfrak{M})^{\frac{1}{2}}$, it can be compared directly with the velocity at the nozzle throat. Thus, for $\gamma = 1.2$, $c^* = 1.5V_{th}$. In general, for a properly designed nozzle, $V_{th} < c^* < V_e$, so that c^* equals the gas velocity at some station in the divergent part of the exhaust nozzle.

The efficiency of the rocket engine can be discussed in terms of the concepts of the ideal rocket motor. Five efficiencies deserve discussion here: combustion efficiency, expansion or "cycle" efficiency, nozzle efficiency, thermal efficiency, and total efficiency. In addition, the attempts to define a so-called propulsive efficiency will be examined.

The *combustion efficiency* η_c is defined as the ratio of the actual enthalpy released by combustion to the ideal enthalpy that would be released if the reaction were to go to completion.

$$\eta_c = \frac{(\Delta h_c + h_i)_{\text{actual}}}{(\Delta h_c + h_i)_{\text{ideal}}} = \frac{(T_c)_{\text{actual}}}{(T_c)_{\text{ideal}}} \tag{2-25}$$

The *ideal expansion or cycle efficiency* η_i expresses the fraction of the enthalpy available in the combustion chamber that can ideally be converted to kinetic energy in the exhaust jet. Let h_i represent the enthalpy of the product gas at the injection temperature with reference to 0°K.

$$\eta_i = \frac{(\frac{1}{2}V_e^2)_{\text{ideal}}}{\Delta h_c + h_i} = \frac{(\frac{1}{2}V_e^2)_{\text{ideal}}}{c_p T_c} \tag{2-26}$$

From Eq. 2-10,

$$\eta_i = 1 - \left(\frac{p_\infty}{p_c}\right)^{\frac{\gamma-1}{\gamma}} \tag{2-27}$$

The *nozzle efficiency* η_n is defined as the ratio of the actual kinetic energy in the exhaust jet to that which could be produced ideally at the specified pressure ratio. (Compare with diffuser efficiency in ramjets, Sec. E.)

$$\eta_n = \frac{(\frac{1}{2}V_e^2)_{\text{actual}}}{(\frac{1}{2}V_e^2)_{\text{ideal}}} = \frac{(T_c - T_e)_{\text{actual}}}{(T_c - T_e)_{\text{ideal}}} \tag{2-28}$$

The *thermal efficiency* η_{th} is the ratio of actual kinetic energy in the exhaust jet to the total enthalpy that could ideally be produced by the combustion reaction.

$$\eta_{th} = \frac{(\frac{1}{2}V_e^2)_{\text{actual}}}{(\Delta h_c)_{\text{ideal}} + h_i} \tag{2-29}$$

Conversion losses in the combustion process and in the expansion process, and the enthalpy discarded in the hot exhaust jet, are all represented in the thermal efficiency.

$$\eta_{th} = \eta_c \eta_n \eta_i = \frac{(V_e^2)_{\text{actual}}}{(V_e^2)_{\max}} \tag{2-30}$$

The four efficiencies mentioned so far refer to the rocket motor simply as a heat engine and do not involve the energy quantities connected with flight. When flight is considered, it is possible to define an *over-all* or a *total efficiency* η_0 as the ratio of propulsive power (output) to the rate of consumption of energy in the rocket motor (input). The output is simply $V_\infty V_e$ per unit mass flow of propellant, where V_∞ is the flight speed. The input is the sum of $(\Delta h_c + h_i)$ plus the kinetic energy $\frac{1}{2}V_\infty^2$ of the propellant. The latter quantity results from the recognition that a quantity of propellant possesses more total energy when it is in motion than when at rest.

$$\begin{aligned}\eta_0 &= \frac{V_\infty V_e}{(\Delta h_c + h_i)_{\text{ideal}} + \frac{1}{2}V_\infty^2} \\ &= \sqrt{\eta_{\text{th}}}\left[\frac{2V_\infty (V_e)_{\max}}{(V_e^2)_{\max} + V_\infty^2}\right]\end{aligned} \tag{2-31}$$

The maximum value of η_0 is $\sqrt{\eta_{\text{th}}}$, and this value is reached when the flight velocity equals the maximum theoretical exhaust velocity. (This result has led to statements in the early literature that a rocket's flight speed could not exceed the maximum theoretical exhaust velocity. Clearly, this conclusion is not justified.)

There have been several attempts in the past to define a suitable "propulsive efficiency" by analogy with the corresponding case of a propeller-driven airplane, but these attempts have always failed because of the lack of a logical definition of mechanical input. If the analogy is to be carried through, the so-called propulsive efficiency will have to satisfy three requirements: (1) it should be a ratio of a mechanical output to a mechanical input; (2) the product of the propulsive efficiency and the thermal efficiency η_{th} should be equal to the total efficiency η_0; and (3) it must under no flight condition exceed unity.

The definition of propulsive efficiency that has been most prominent in the literature is

$$\eta_p = \frac{\text{output}}{\text{input}} = \frac{\text{output}}{\text{output} + \text{loss}} = \frac{V_\infty V_e}{V_\infty V_e + \frac{1}{2}(V_e - V_\infty)^2} = \frac{2V_\infty V_e}{(V_\infty^2 + V_e^2)}$$

The denominator is supposed to represent the sum of the propulsive work and the absolute kinetic energy of the jet (a loss). The present author objects to this propulsive efficiency on the grounds that it offers no clear definition of mechanical input and that, even if this particular definition were allowed, it fails to meet the requirement that $\eta_p \eta_{\text{th}} = \eta_0$.[1]

As a conclusion to this discussion of efficiency, it may be remarked that the concept of efficiency has not been at all useful in the field of rocket engines. Rockets are always compared on the basis of I_{sp} or c^* values, or on specific propellant consumption, and rarely is efficiency

[1] Editor's note: The author would find the same difficulty with any engine whenever the kinetic energy of the fuel is included.